Indian Statistical Institute
Second Semester Examination 2005-2006
M.Math I Year
Fourier Analysis

Time: 3 hrs   Date: 03-05-2006   Maximum Marks: 60

Answer as many questions as you can. You may use your class- room notes.

You may assume the following result stated in class: If S is a tempered distribution such that supp $\hat{S} = \{0\}$, then S is a polynomial.

1. Let f be a 2π-periodic function whose Fourier coefficients $\hat{f}(n)$ are given by $\hat{f}(n) = e^{-\alpha[\log|n|]^2}$, $n \neq 0$, where $\alpha > 0$ is a constant.

   a) Is f square-integrable on $[-\pi, \pi]$?

   b) Is f a C^∞-function?

   Justify your answers.

2. Let E be a set of finite positive measure in $I\!R$. Prove that $\hat{\chi}_E$ cannot be integrable on $I\!R$.

3. Let f be a continuous function on the real line with $|f(x)| \leq e^{|x|}\, e^{-|x|^3}$, $x \in I\!R$. Prove that

   a) $f \in L^1(I\!R)$

   b) If $|\hat{f}(\lambda)| \leq B\, e^{-\beta\lambda^2}$ for some $B, \beta > 0$, prove that $f \equiv 0$.

4. Let $f(x) = e^{-\frac{1}{x^2}} e^{-|x|}$, $x \neq 0$ & $f(0) = \lim_{x \to 0} f(x)$. Prove that $\hat{f} \in \mathcal{S}(I\!R)$.

5. Let $f(x) = xe^{-x^2}$. If g is a function which is in $L^p \cap L^\infty$ for some $1 \leq p < \infty$ and $f * g \equiv 0$, show that $g = 0$ a.e.

6. Let $E \subseteq I\!R^3$ be the set $\{x \in I\!R^3 : \|x\| \leq 1\}$. Calculate $\hat{\chi}_E$. Hint: $\hat{\chi}_E$ is a radial function. So calculate it at $(0, 0, r)$ using polar coordinates.

7. Let $f(x) = \chi_{[-\infty,1]} e^{x^2}$. What is T_f'?